LETTER TO THE EDITOR

Exactly solvable quantum spin tubes and ladders

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Abstract. We find families of integrable n-leg spin-$\frac{1}{2}$ ladders and tubes with general isotropic exchange interactions between spins. These models are equivalent to $su(N)$ spin chains with $N = 2^n$. Arbitrary rung interactions in the spin tubes and ladders induce chemical potentials in the equivalent spin chains. The potentials are n-dependent and differ for the tube and ladder models. The models are solvable by means of nested Bethe ansatze.

The physics of quantum spin ladders has attracted a great deal of recent theoretical and experimental interest. It is now well established that the properties of n-leg Heisenberg spin-$\frac{1}{2}$ ladders show a remarkable n-dependence [1]. For n odd the Heisenberg ladders have a gapless ground state with a quasi long-range order, while for n even there is a spin liquid ground state with short-range correlations and an energy gap. A number of different compounds have been found which confirm this behaviour.

Of course, for $n = 1$ these properties are borne out by the solvable Heisenberg chain. However, the more general n-leg Heisenberg ladders defy an exact solution. Nevertheless, some solvable spin ladders have been found. In recent progress, Wang [2] discussed a 2-leg ladder with Hamiltonian

$$H = \tfrac{1}{4} \sum_{j=1}^{L} (\boldsymbol{\sigma}_j^{(1)} \cdot \boldsymbol{\sigma}_{j+1}^{(1)} + \boldsymbol{\sigma}_j^{(2)} \cdot \boldsymbol{\sigma}_{j+1}^{(2)}) + \tfrac{1}{2} J \sum_{j=1}^{L} \boldsymbol{\sigma}_j^{(1)} \cdot \boldsymbol{\sigma}_j^{(2)} + \tfrac{1}{4} \sum_{j=1}^{L} (\boldsymbol{\sigma}_j^{(1)} \cdot \boldsymbol{\sigma}_{j+1}^{(1)})(\boldsymbol{\sigma}_j^{(2)} \cdot \boldsymbol{\sigma}_{j+1}^{(2)})$$

(1)

where $\boldsymbol{\sigma}_j^{(1)}$ and $\boldsymbol{\sigma}_j^{(2)}$ are Pauli matrices acting at site j on legs 1 and 2 of the ladder, with $\boldsymbol{\sigma}_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z)$. L is the number of rungs and periodic boundary conditions are imposed. It is to be noted that the last term of the Hamiltonian defines biquadratic interactions which couple sites on the two legs of the ladder. Such interchain coupling can be of experimental importance [2]. The variable parameter J measures the strength of the rung interactions. Another, more complicated, though with no variable parameter, solvable 2-leg Hamiltonian has been introduced by Albeverio and Fei [3], about which we will say more elsewhere. Yet other 2-leg Hamiltonians have been defined with special matrix-product ground states [4]. Our interest here is in integrable or exactly solvable models with an underlying R-matrix.

The solvability of Wang's model lies in the observation that it can be mapped to the Hamiltonian of an $su(4)$-invariant spin chain for $J = 0$, whilst for $J \neq 0$ the rung interactions

take the form of a chemical potential. Specifically, Hamiltonian (1) can be written [2]

$$H = \tfrac{1}{4} \sum_{j=1}^{L} (1 + \boldsymbol{\sigma}_j^{(1)} \cdot \boldsymbol{\sigma}_{j+1}^{(1)})(1 + \boldsymbol{\sigma}_j^{(2)} \cdot \boldsymbol{\sigma}_{j+1}^{(2)}) + \tfrac{1}{2} J \sum_{j=1}^{L} (\boldsymbol{\sigma}_j^{(1)} \cdot \boldsymbol{\sigma}_j^{(2)} - 1) + \tfrac{1}{2}(J - \tfrac{1}{2})L$$

$$= \sum_{j=1}^{L} \sum_{\alpha,\beta=1}^{4} X_j^{\alpha\beta} X_{j+1}^{\beta\alpha} - 2J \sum_{j=1}^{L} X_j^{11} \tag{2}$$

up to an irrelevant constant. The operators $X_j^{\alpha\beta} = |\alpha_j\rangle\langle\beta_j|$, where $|\alpha_j\rangle$ are the (orthogonalized) eigenstates of the single-rung Hamiltonian. The constant $2J$ in the last term indicates a chemical potential applied on N_1 where, in general, the operators $N_\alpha = \sum_{j=1}^{N} X_j^{\alpha\alpha}$ are conserved quantities.

The underlying integrability of the 2-leg ladder Hamiltonian (1) is thus seen to be due to the known R-matrix associated with $su(4)$ [5]. The key ingredient is the permutation operator

$$P_{j,j+1} = \tfrac{1}{4}(1 + \boldsymbol{\sigma}_j^{(1)} \cdot \boldsymbol{\sigma}_{j+1}^{(1)})(1 + \boldsymbol{\sigma}_j^{(2)} \cdot \boldsymbol{\sigma}_{j+1}^{(2)}) = \sum_{\alpha,\beta=1}^{4} X_j^{\alpha\beta} X_{j+1}^{\beta\alpha}. \tag{3}$$

Our starting point is to note that for an n-leg ladder the permutation operator can be written

$$P_{j,j+1} = \frac{1}{N} \prod_{i=1}^{n} (1 + \boldsymbol{\sigma}_j^{(i)} \cdot \boldsymbol{\sigma}_{j+1}^{(i)}) = \sum_{\alpha,\beta=1}^{N} X_j^{\alpha\beta} X_{j+1}^{\beta\alpha} \tag{4}$$

in which $X_j^{\alpha\beta}$ are $N \times N$ $su(N)$ operators with $N = 2^n$. Thus, we can immediately write down a family of solvable n-leg ladders with Hamiltonian

$$H_n^{\text{ladder}} = \frac{1}{2^n} \sum_{j=1}^{L} \prod_{i=1}^{n} (1 + \boldsymbol{\sigma}_j^{(i)} \cdot \boldsymbol{\sigma}_{j+1}^{(i)}). \tag{5}$$

The operators $\boldsymbol{\sigma}_j^{(i)}$ are defined on leg i. The 3-leg Hamiltonian reads

$$H_3^{\text{ladder}} = \tfrac{1}{8} \sum_{j=1}^{L} [\boldsymbol{\sigma}_j^{(1)} \cdot \boldsymbol{\sigma}_{j+1}^{(1)} + \boldsymbol{\sigma}_j^{(2)} \cdot \boldsymbol{\sigma}_{j+1}^{(2)} + \boldsymbol{\sigma}_j^{(3)} \cdot \boldsymbol{\sigma}_{j+1}^{(3)}$$

$$+ (\boldsymbol{\sigma}_j^{(1)} \cdot \boldsymbol{\sigma}_{j+1}^{(1)})(\boldsymbol{\sigma}_j^{(2)} \cdot \boldsymbol{\sigma}_{j+1}^{(2)}) + (\boldsymbol{\sigma}_j^{(1)} \cdot \boldsymbol{\sigma}_{j+1}^{(1)})(\boldsymbol{\sigma}_j^{(3)} \cdot \boldsymbol{\sigma}_{j+1}^{(3)})$$

$$+ (\boldsymbol{\sigma}_j^{(2)} \cdot \boldsymbol{\sigma}_{j+1}^{(2)})(\boldsymbol{\sigma}_j^{(3)} \cdot \boldsymbol{\sigma}_{j+1}^{(3)}) + (\boldsymbol{\sigma}_j^{(1)} \cdot \boldsymbol{\sigma}_{j+1}^{(1)})(\boldsymbol{\sigma}_j^{(2)} \cdot \boldsymbol{\sigma}_{j+1}^{(2)})(\boldsymbol{\sigma}_j^{(3)} \cdot \boldsymbol{\sigma}_{j+1}^{(3)})] + \tfrac{1}{8}L. \tag{6}$$

In general, this family of n-leg ladders includes up to n-body interactions. As the number of legs increases these interactions become increasingly nonlocal with, for example, $\boldsymbol{\sigma}_j^{(1)} \cdot \boldsymbol{\sigma}_{j+1}^{(1)}$ on leg 1 interacting with $\boldsymbol{\sigma}_j^{(n)} \cdot \boldsymbol{\sigma}_{j+1}^{(n)}$ on leg n. However, such interactions are necessary to preserve the integrability of the model†.

To some extent, the nonlocality can be overcome by considering quantum spin 'tubes', rather than spin ladders. The 3- and 4-tubes are depicted in figure 1. The n-leg ladder Hamiltonian (5) applies equally well to the n-tube. In this way, the above Hamiltonian H_3^{ladder} is equal to H_3^{tube}. The eigenspectra of the Hamiltonians H_n^{ladder} and H_n^{tube} are equivalent to that of the $su(N)$ permutation Hamiltonian [6], with eigenvalues

$$E = L - \sum_{j=1}^{M_1} \frac{1}{(\lambda_j^{(1)})^2 + \tfrac{1}{4}} \tag{7}$$

† The situation is somewhat akin to the price paid in the integrable spin-S generalizations of the Heisenberg chain for which terms up to order $(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^{2S}$ appear [5].
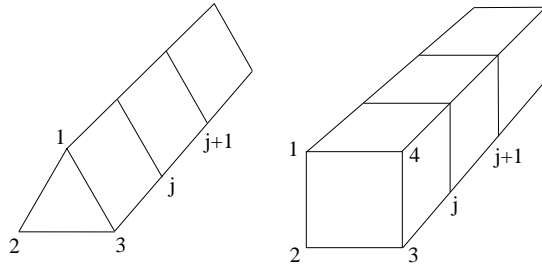
Figure 1. Labelling of legs and rungs on the 3- and 4-leg spin tubes.

where we recall that $N = 2^n$. The $su(N)$ Bethe equations are given in terms of $N - 1$ roots $\lambda_j^{(r)}$

$$
\left(\frac{\lambda_j^{(1)} - \frac{i}{2}}{\lambda_j^{(1)} + \frac{i}{2}}\right)^L = \prod_{k \neq j}^{M_1} \frac{\lambda_j^{(1)} - \lambda_k^{(1)} - i}{\lambda_j^{(1)} - \lambda_k^{(1)} + i} \prod_{k=1}^{M_2} \frac{\lambda_j^{(1)} - \lambda_k^{(2)} + \frac{i}{2}}{\lambda_j^{(1)} - \lambda_k^{(2)} - \frac{i}{2}}
$$
$$
\prod_{k \neq j}^{M_r} \frac{\lambda_j^{(r)} - \lambda_k^{(r)} - i}{\lambda_j^{(r)} - \lambda_k^{(r)} + i} = \prod_{k=1}^{M_{r-1}} \frac{\lambda_j^{(r)} - \lambda_k^{(r-1)} - \frac{i}{2}}{\lambda_j^{(r)} - \lambda_k^{(r-1)} + \frac{i}{2}} \prod_{k=1}^{M_{r+1}} \frac{\lambda_j^{(r)} - \lambda_k^{(r+1)} - \frac{i}{2}}{\lambda_j^{(r)} - \lambda_k^{(r+1)} + \frac{i}{2}}
\tag{8}
$$

where $j = 1, \ldots, M_r$ with $r = 1, \ldots, N - 1$. We take $M_N = 0$. It is well known that the isotropic $su(N)$ models are critical with no gap.

Note that the permutation operator for the $su(N)$ models can be written in terms of spin-S operators as

$$
P_{j,j+1} = (-)^{2S} \sum_{i=0}^{2S} (-)^i \prod_{k \neq i}^{2S} \frac{Y_j - x_k}{x_i - x_k}
\tag{9}
$$

where $Y_j = \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$, $x_k = \frac{1}{2}k(k+1) - S(S+1)$ and $N = 2S + 1$. This gives an equivalent, $su(2)$-invariant, representation of the eigenspectrum. The Hamiltonian of the n-leg spin ladder or spin tube is of size $2^{nL} \times 2^{nL}$, which is equivalent to that of the $su(N)$ chain of length L, namely $N^L \times N^L$.

Rung interactions of variable strength can also be introduced in these models. For the ladder and tube Hamiltonians, we define

$$
H_n^{\text{ladder}}(J) = \sum_{j=1}^{L} \left[\frac{1}{2^n} \prod_{i=1}^{n} (1 + \boldsymbol{\sigma}_j^{(i)} \cdot \boldsymbol{\sigma}_{j+1}^{(i)}) + \frac{1}{2} J \sum_{i=1}^{n-1} (\boldsymbol{\sigma}_j^{(i)} \cdot \boldsymbol{\sigma}_j^{(i+1)} - 1) \right]
\tag{10}
$$

$$
H_n^{\text{tube}}(J) = \sum_{j=1}^{L} \left[\frac{1}{2^n} \prod_{i=1}^{n} (1 + \boldsymbol{\sigma}_j^{(i)} \cdot \boldsymbol{\sigma}_{j+1}^{(i)}) + \frac{1}{2} J \sum_{i=1}^{n} (\boldsymbol{\sigma}_j^{(i)} \cdot \boldsymbol{\sigma}_j^{(i+1)} - 1) \right]
\tag{11}
$$

where for the tube we have $\sigma_j^{(n+1)} = \sigma_j^{(1)}$. The equivalent Hamiltonians can be constructed from a consideration of the rung basis states. The first term in equations (10) and (11) are simply permutation operators. The second terms can also be expressed in terms of X operators, using the fact that any Hamiltonian h may be expressed as $h = \sum_i \lambda_i |\alpha_i\rangle\langle\alpha_i| = \sum_i \lambda_i X^{ii}$. Here λ_i is the eigenvalue corresponding to eigenstate $|\alpha_i\rangle$ and the sum is over all eigenstates. We choose to write $h = \sum_i (\lambda_i - \lambda_{\max}) X^{ii}$, where $\lambda_{\max}$ is the largest eigenvalue. This adds a constant to the Hamiltonian but does not change the underlying physics. It has the advantage of leading to a more compact representation of h in terms of X matrices. We list here the first

few cases:

$$H_3^{\text{ladder}}(J) = \sum_{j=1}^{L} \sum_{\alpha,\beta=1}^{8} X_j^{\alpha\beta} X_{j+1}^{\beta\alpha} - J \sum_{j=1}^{L} (X_j^{11} + 3X_j^{22} + 3X_j^{33} + X_j^{44}) \tag{12}$$

$$H_3^{\text{tube}}(J) = \sum_{j=1}^{L} \sum_{\alpha,\beta=1}^{8} X_j^{\alpha\beta} X_{j+1}^{\beta\alpha} - 3J \sum_{j=1}^{L} (X_j^{11} + X_j^{22} + X_j^{33} + X_j^{44}) \tag{13}$$

$$H_4^{\text{tube}}(J) = \sum_{j=1}^{L} \sum_{\alpha,\beta=1}^{16} X_j^{\alpha\beta} X_{j+1}^{\beta\alpha} - 2J \sum_{j=1}^{L} (3X_j^{11} + 2X_j^{22} + 2X_j^{33} + 2X_j^{44} + X_j^{55}$$
$$+ X_j^{66} + X_j^{77} + X_j^{88} + X_j^{99} + X_j^{10\,10} + X_j^{11\,11}). \tag{14}$$

These models with rung-mediated chemical potentials can again be solved via nested Bethe ansatze. For example, for the 3-tube we find

$$E_3^{\text{tube}}(J) = L(1 - 3J) - \sum_{j=1}^{M_1} \left(\frac{1}{(\lambda_j^{(1)})^2 + \frac{1}{4}} - 3J \right) \tag{15}$$

where the roots $\lambda_j^{(1)}$ are given by the Bethe equations (8) with $N = 8$.

Our results can be extended in a number of directions. Wang also introduced a solvable 2-leg ladder based on the supersymmetric permutation operator. The family of supersymmetric models will clearly lead to other solvable n-leg ladders and n-tubes. Both of the solvable 2-leg ladders found by Wang have interesting physics, with a transition to a rung-dimerized phase with a spin gap [2]. The physics of the solvable models presented here is expected to be of equal interest, about which we hope to report in the near future.

References

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